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ON p-HARMONIC FUNCTIONS IN THE COMPLEX PLANE AND CURVATURE

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ABSTRACT
In the complex plane the p-harmonic equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u)=0$, $1<p<\infty$, exhibits some features reminiscent of Function Theory. Our results about curvature in this structure complement known facts about minimal surfaces and harmonic functions. Quasiregular mappings are used.

1. Introduction

In the complex plane *the p-harmonic equation*

$$\frac{\partial}{\partial x}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial y}\right)=0 \tag{1.1}$$

exhibits some special features reminiscent of Function Theory. Here $1<p<\infty$ and (1.1) reduces to the Laplace equation $\Delta u=0$, when $p=2$. The solutions are called *p-harmonic functions* and they are real-analytic except possibly at isolated points.

This paper is devoted to the study of curvature in a structure induced by p-harmonic functions. Our results liken known facts about minimal surfaces and about harmonic functions.

Given a p-harmonic function u in a simply connected domain $G\subset\mathbf{C}$, one can construct a *conjugate function* v that is q-harmonic in G, $1/p+1/q=1$, and $\nabla u\cdot\nabla v=0$. Moreover,

$$F=u+iv$$

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admits a *Stoïlow representation*. (See Section 3 for this kind of conjugation.) Note that the orthogonality $\nabla u \cdot \nabla v = 0$ means that the level lines of u are orthogonal to those of v. This is the motivation for our study of the level lines of p-harmonic functions. If k denotes the curvature for the level lines of u and $-h$ the same quantity with respect to the conjugate function v, then $\bar{\varphi} = k - ih$ is locally integrable over G to any power $s < 2$. This result (Theorem 4.11) is sharp even in the classical case, when $u + iv$ is holomorphic.

In Section 5 we study the Gauss curvature K for the corresponding p-harmonic surfaces "$x_3 = u(x_1, x_2)$" in $\mathbf{R}^3$. Minimal surfaces and harmonic surfaces ($p = 2$) have the property that $K \leqq 0$ and $K = 0$ only at isolated points, unless the surface is a plane. Even p-harmonic surfaces obey this rigidity (Theorem 5.3).

Entire p-harmonic surfaces (the corresponding p-harmonic functions are defined in the whole complex plane) have total curvature $\leqq -2\pi$ or $=0$ (Theorem 5.5). Only the planes have total curvature zero among the p-harmonic surfaces. The gap $(-2\pi, 0)$ reflects a fascinating phenomenon. Our proof is based upon the Picard theorem for quasiregular mappings.

In Sections 2 and 3 we have, for the benefit of the reader, assembled necessary preliminaries.

2. Quasiregular mappings and p-harmonic functions

To be on the safe side we clarify the concept of solutions. A function u in the local Sobolev space $W^{1,p}_{\text{loc}}(G)$, G being a domain in $\mathbf{R}^n$, and $1 < p < \infty$, is called *p-harmonic* in G, if

$$\int_G |\nabla u|^{p-2} \nabla u \cdot \nabla \eta \, dx = 0 \tag{2.1}$$

whenever $\eta \in C_0^\infty(G)$. This is the weak form of the equation $\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$. It is known that u is continuous (after a redefinition in a set of measure zero) and, indeed, even that ∇u is continuous. Actually, u is of class $C^{1,\alpha}_{\text{loc}}(G)$ and the second derivatives exist in Sobolev's sense, cf. [D], [E], [Le 3], [To], and [U].

From now on we confine the discussion to the complex plane. In the non-linear case $p \neq 2$ the p-harmonic equation

$$|\nabla u|^2 \Delta u + \frac{p-2}{2} \nabla u \cdot \nabla |\nabla u|^2 = 0 \tag{2.2}$$

degenerates at points where $\nabla u = 0$. A way of handling this difficulty was proposed by B. Bojarski and T. Iwaniec, who observed that the complex gradient $f = u_x - iu_y$ of a p-harmonic function u is quasiregular [B–I 1]. (An analogous situation is encountered in connection with uniformly subsonic gas flows [Be, Section 8].) Another proof for the quasiregularity has been constructed by J. Manfredi, cf. [M]. See also [A–L, Theorem 1]. A slightly different approach can be found in [Al 1].

2.3. THEOREM. *(The continuous representative for) the complex gradient $f = u_x - iu_y$ of a p-harmonic function u in G is quasiregular. More precisely*

(i) *f is continuous in G,*

(ii) *the Sobolev derivatives $f_{\bar z}$ and f_z exist and belong to $L^2_{\mathrm{loc}}(G)$, and*

(iii) *$|f_{\bar z}| \leqq |1 - 2/p|\,|f_z|$ a.e. in G.*

It is essential that $|1 - 2/p| < 1$. For the theory of quasiregular mappings in the plane we refer the reader to [Bo], [Re], [V]. An important consequence of the quasiregularity is that f admits a *Stoïlow representation* $f = h \circ \xi$, ξ being quasiconformal in G (ξ is a homeomorphism satisfying the same conditions (i), (ii), and (iii) as f) and h being holomorphic in the domain $\xi(G)$. This implies that the zeros of f are isolated points unless u is constant, i.e., *the singular set*

$$S = \{z \in G \mid f(z) = 0\} \tag{2.4}$$

is discrete. The formula

$$f_{\bar z} = \frac{2-p}{2p}\left\{\frac{\bar f}{f} f_z + \frac{f}{\bar f}\overline{f_z}\right\} \tag{2.5}$$

holds outside S [B–I 1], and for $p = 2$ we have merely the Cauchy–Riemann equations $f_{\bar z} = 0$.

Outside the singular set u is even real-analytic [Le 1, p. 208]. There is a striking phenomenon in the non-linear case $p \neq 2$: if $f(z) = 0$ at some $z \in G$ and if u is real-analytic in a neighborhood of z, then u reduces to a constant [Le 2]. This is not true for harmonic functions ($p = 2$)!

Note that (iii) can be written as

$$|f_z|^2 + |f_{\bar z}|^2 \leqq \left(p - 1 + \frac{1}{p-1}\right)(|f_z|^2 - |f_{\bar z}|^2). \tag{2.6}$$

The Jacobian $J_f = |f_z|^2 - |f_{\bar z}|^2$ of f is real-analytic in $G \setminus S$ and its zeros in

$G \setminus S$ are, indeed, isolated, unless u reduces to a linear function [A–L, Theorem 3]. This yields interesting information about the Gauss curvature (5.2) of a p-harmonic surface.

According to an advanced theory due to Yu. G. Reshetnyak the coordinate functions of a quasiregular mapping are free extremals for a variational integral, the integrand of which, however, depends on the quasiregular mapping itself (Dirichlet's principle). The corresponding Euler–Lagrange equation is elliptic and in the two-dimensional case it is also linear. See [R 2] or [B–I 2] for the details. Moreover, the logarithm of the Euclidean norm of the mapping satisfies the same Euler–Lagrange equation as the coordinate functions [R 2]. See also [B–I 2, Lemma 6.2] and the presentation in [R 3].

From our point of view the variational approach sketched above has the consequence for a p-harmonic function u that

$$v = \log \frac{1}{|f|} \qquad (f = u_x - iu_y) \tag{2.7}$$

is a solution to the linear Euler–Lagrange equation induced by f, except at the singular points. In the language of [G–L–M] and [L–M] v is a super-extremal in G (v is a lower semicontinuous function obeying the comparison principle in G with respect to the solutions of the Eular–Lagrange equation).

If f is not identically zero, then $\log|f|$ is locally integrable in G to any finite power [L–M, Lemma 2.22]. A good example is the harmonic function $u = x^2 - y^2$; $\log|f| = \log|2z|$. The behaviour of f near its zeros is reflected in the sharp result below. (The analogous local integrability has been studied in [Li] for p-superharmonic functions and their derivatives, but the above v is not exactly p-superharmonic, when $p \neq 2$.)

2.8. Theorem. *Suppose that u is p-harmonic in G and that $f = u_x - iu_y$ is not identically zero. Then*

$$\int\!\!\int_D |\nabla \log |f||^{2-\varepsilon} dx\,dy < \infty \qquad (0 < \varepsilon \leqq 2)$$

whenever $0 < \varepsilon \leqq 2$ and D has compact closure in G.

Proof. In view of Reshetnyak's theorem [B–I 2, Lemma 6.2] this is merely a special case of [L–M, Theorem 2.24]. □

To make a long story short the local integrability result expressed in

Theorem 2.8 is all we need from the variational approach to quasiregular mappings.

3. Conjugate functions

The purpose of the background given in this section is to motivate the study of level curves.

If u is p-harmonic in a simply connected domain G then there is a function v (unique up to a constant) such that

$$v_x = -|\nabla u|^{p-2}u_y, \qquad v_y = |\nabla u|^{p-2}u_x \tag{3.1}$$

in G. For $p = 2$ these are the Cauchy–Riemann equations. The function v is q-harmonic in G, q being the conjugate exponent to p:

$$1/p + 1/q = 1.$$

From our point of view a most interesting feature is that

$$\nabla u \cdot \nabla v = 0, \tag{3.2}$$

i.e., the level curves of u and of v are orthogonal to each other apart from the singular set S. In multiply connected domains the conjugation described above is possible at least locally. A good example in $0 < |z| < \infty$ is

$$u + iv = \frac{p-1}{p-2}|z|^{(p-2)/(p-1)} + i \arg z \qquad (p \neq 2) \tag{3.3}$$

and $\log z$ ($p = 2$). See [A–L, §3] for this kind of function theory.

The mapping

$$F = u + iv$$

is interior in the sense of Stoïlow: it admits the representation $F = H \circ \chi$, H being a holomorphic function and χ a topological mapping of G [A–L, Theorem 5]. Hence the sets

$$\{(x, y) \in G \mid u(x, y) = t\}$$

usually represent genuine level lines. In the harmonic case ($p = q = 2$) F itself is holomorphic, but for $p \neq 2$ F is not even quasiregular in $G \setminus S$, if $S \neq \varnothing$. However, F is locally quasiregular in $G \setminus S$. (This means that the restriction of F to any domain D with compact closure in $G \setminus S$ is quasiregular.)

Although the level curves of F intersect at right angles in $G \setminus S$, other angles are distorted, when $p \neq 2$. The Jacobian for $F = u + iv$ is

$$\frac{\partial(u, v)}{\partial(x, y)} = |\nabla u|^p = |\nabla v|^q \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right) \tag{3.4}$$

and it is zero precisely in the singular set S.

The system (3.1) can be written in the form

$$g = -i|f|^{p-2}f \tag{3.5}$$

where $f = u_x - iu_y$ and $g = v_x - iv_y$. The orthogonality (3.2) between level lines is expressed by the relation $f\bar{g} + \bar{f}g = 0$. We cannot resist mentioning that

$$f_{\bar{z}} = \nu f_z + \overline{\nu f_z}, \qquad g_{\bar{z}} = \nu g_z + \overline{\nu g_z}$$

with the same ν in both equations:

$$\nu = \left(\frac{1}{p} - \frac{1}{2}\right)\frac{\bar{f}}{f} = \left(\frac{1}{q} - \frac{1}{2}\right)\frac{\bar{g}}{g},$$

cf. (2.5).

4. The curvature of the level lines

For a sufficiently regular function u the curvature of the level lines "$u(x, y) = \text{Constant}$" is given by

$$k = -\frac{u_y^2 u_{xx} - 2u_y u_x u_{xy} + u_x^2 u_{yy}}{|\nabla u|^3}, \tag{4.1}$$

and

$$h = \frac{(u_{xx} - u_{yy})u_x u_y - u_{xy}(u_x^2 - u_y^2)}{|\nabla u|^3} \tag{4.2}$$

represents the curvature of the trajectories orthogonal to the level lines. These formulae can be found in ordinary books on Differential Geometry. Following G. Talenti [T2] we define $\varphi = k + ih$ and write (4.1) and (4.2) as

$$\varphi = k + ih = -2\frac{\partial}{\partial z}\left(\frac{\bar{f}}{|f|}\right), \qquad f = u_x - iu_y. \tag{4.3}$$

We are interested in the case when u is p-harmonic in a domain G. Let v be the conjugate q-harmonic function defined (at least locally) by (3.1). Suppose that $F = u + iv$ is not a linear function, so that (4.3) makes sense except possibly at isolated points. Outside these singular points φ is even real-analytic. By (3.5) we have

$$i\varphi = -2\frac{\partial}{\partial z}\left(\frac{g}{|g|}\right), \qquad g = v_x - iv_y, \tag{4.4}$$

so that $-h$ represents the curvature of the level lines for v. This "orientation" makes

$$\frac{\varphi}{|f|} = \frac{\partial}{\partial z}\left(-\frac{1}{f}\right)$$

into a meromorphic function in the classical case ($p = 2$) and hence $k/|f|$ and $h/|f|$ are themselves conjugate harmonic functions in $G\setminus S$, when $u + iv$ is holomorphic. See [T2, Theorem 3] for results of this kind.

However, such a nice conjugation would be too much to ask for in the general case. We have the following counterpart to the classical formula above.

4.5. THEOREM. *If u is p-harmonic, then*

$$\frac{\varphi}{|f|} = \frac{\partial}{\partial z}\left(-\frac{1}{f}\right) + \frac{p-2}{p}\operatorname{Re}\frac{\partial}{\partial z}\left(-\frac{1}{f}\right)$$

when $f = u_x - iu_y \neq 0$.

PROOF. Evaluating (4.3) we have

$$\varphi = |f|\left\{\frac{1}{f^2}\frac{\partial f}{\partial z} - \frac{1}{|f|^2}\frac{\partial \bar{f}}{\partial z}\right\}$$

and using (2.5) and the rule $\partial\bar{f}/\partial z = \overline{\partial f/\partial\bar{z}}$ we arrive at

$$\varphi = |f|\left\{\frac{1}{f^2}\frac{\partial f}{\partial z} - \frac{2-p}{2p}\left[\frac{\partial}{\partial z}\left(-\frac{1}{f}\right) + \overline{\frac{\partial}{\partial z}\left(-\frac{1}{f}\right)}\right]\right\}.$$

This gives the desired formula. □

4.6. LEMMA. *Suppose that u is p-harmonic in G. Then*

$$|\varphi|^2 = \left|\frac{\partial \ln(f\bar{f})}{\partial z}\right|^2 + \frac{4p}{p-2}\left|\frac{1}{f}\frac{\partial f}{\partial \bar{z}}\right|^2 \qquad (p \neq 2) \tag{4.7}$$

in $G\setminus S$. Here $f = u_x - iu_y$.

PROOF. We have

$$\frac{\partial}{\partial z}\ln(f\bar{f})=\frac{1}{f}\frac{\partial f}{\partial z}+\frac{1}{\bar{f}}\frac{\partial \bar{f}}{\partial z}$$

and according to (4.3)

$$\frac{f\varphi}{|f|}=\frac{1}{f}\frac{\partial f}{\partial z}-\frac{1}{\bar{f}}\frac{\partial \bar{f}}{\partial z}.$$

Hence

$$\varphi\bar{\varphi}=\frac{f\varphi}{|f|}\cdot\frac{\overline{f\varphi}}{|f|}=\left|\frac{\partial}{\partial z}\ln(f\bar{f})\right|^2-2\frac{\partial f}{\partial \bar{z}}\left\{\frac{1}{f^2}\frac{\partial f}{\partial z}+\frac{1}{\bar{f}^2}\overline{\frac{\partial f}{\partial z}}\right\}$$

$$=\left|\frac{\partial}{\partial z}\ln(f\bar{f})\right|^2-2|f|^{-2}\frac{\partial f}{\partial \bar{z}}\left\{\frac{\bar{f}}{f}\frac{\partial f}{\partial z}+\frac{f}{\bar{f}}\overline{\frac{\partial f}{\partial z}}\right\}$$

where we have used $\partial\bar{f}/\partial z=\overline{\partial f/\partial\bar{z}}=\partial f/\partial\bar{z}$. Now (2.5) yields (4.7). □

For $p=2$ we have

$$|\varphi|=\left|\frac{\partial}{\partial z}\ln(f\bar{f})\right|.$$

For $p\geqq 2$, (4.7) shows that

$$|\varphi|\geqq\left|\frac{\partial}{\partial z}\ln(f\bar{f})\right|.$$

For the conjugate q-harmonic function we have, by (4.4) and (4.7),

$$\varphi\bar{\varphi}=i\varphi\overline{(i\varphi)}=\left|\frac{\partial}{\partial z}\ln(g\bar{g})\right|^2+\frac{4q}{q-2}\left|\frac{1}{g}\frac{\partial g}{\partial z}\right|^2$$

and hence (now $q<2$)

$$|\varphi|\leqq\left|\frac{\partial}{\partial z}\ln(g\bar{g})\right|^2.$$

By (3.5) $\ln(g\bar{g})=(p-1)\ln(f\bar{f})$. Collecting results we have

$$\left|\frac{\partial}{\partial z}\ln(f\bar{f})\right|\leqq|\varphi|\leqq\left|\frac{\partial}{\partial z}\ln(g\bar{g})\right|\leqq(p-1)\left|\frac{\partial}{\partial z}\ln(f\bar{f})\right|, \tag{4.8}$$

when $p\geqq 2$.

4.9. THEOREM. *Suppose that the p-harmonic function u and the q-harmonic function v are conjugate in G. Then* (4.8) *holds for* $f = u_x - iu_y$ *and* $g = v_x - iv_y$, *when* $p \geqq 2$. *The inequalities* (4.8) *are reversed, when* $1 < p \leqq 2$.

4.10. REMARK. If u is p-harmonic in a multiply connected domain, then the q-harmonic conjugate can be constructed locally. Hence, by (4.8),

$$\left|\frac{\partial}{\partial z}\ln(f\bar{f})\right| \leqq |\varphi| \leqq (p-1)\left|\frac{\partial}{\partial z}\ln(f\bar{f})\right|$$

globally in G, when $p \geqq 2$. The inequalities are reversed, when $1 < p \leqq 2$.

Combining Remark 4.10 and Theorem 2.8 we obtain a local integrability result for $\varphi = k + ih$.

4.11. THEOREM. *Suppose that u is p-harmonic in G. Then the integral*

$$\int\int_D |\varphi|^{2-\varepsilon} dx\, dy < \infty \qquad (0 < \varepsilon \leqq 2)$$

taken over any domain D with compact closure in G, converges, whenever $0 < \varepsilon \leqq 2$.

As an application we mention that Theorem 4.11 can be used to estimate the length of a level line. If $B \subset\subset G$ is a disk of radius r, then the quantity

$$2\pi r + \int\int_B |k|\, dx\, dy \tag{4.12}$$

majorizes the length of that part of any level line of u that is in B. The Stoïlow representation for $u + iv$ allows us to "integrate by parts" as in [Al 2, p. 2] to obtain (4.12). If r is small, then

$$2\pi r + \pi^{\alpha/2} r^{\alpha} \left\{\int\int_B |\varphi|^{2/(2-\alpha)} dx\, dy\right\}^{1-\alpha/2} \qquad (0 \leqq \alpha < 1)$$

is a good upper bound for (4.12), whenever $\alpha < 1$.

5. The Gauss curvature of a p-harmonic surface

If u is p-harmonic in $G \subset \mathbf{R}^2$, then $x_3 = u(x_1, x_2)$ can be viewed as a representation for a p-harmonic surface in $\mathbf{R}^3$. Its Gauss curvature is

$$K = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2}. \tag{5.1}$$

In complex notation we have

$$K = -\frac{J_f}{(1 + |f|^2)^2} = -\frac{|f_z|^2 - |f_{\bar{z}}|^2}{(1 + |f|^2)^2}, \tag{5.2}$$

where J_f denotes the Jacobian of $f = u_x - iu_y$. The above formulae are well defined in $G \setminus S$, S denoting the singular set. (In the linear case $p = 2$, K is defined by (5.1) at every point. It stands to reason that "the right value" of K is zero at the singular points, when $p \neq 2$. Since the singular set is discrete for non-constant p-harmonic functions, we shall not pursue this question any further.)

The counterpart to the theorem below is fundamental in the theory of minimal surfaces [0, p. 76] and the theorem is known for harmonic surfaces [T 1, p. 2].

5.3. THEOREM. *The Gauss curvature*

$$K \leqq 0$$

for a p-harmonic surface. Either $K = 0$ at most at isolated points, or the surface is a plane.

PROOF. The quasiregularity of f (Theorem 2.3) implies that $J_f \geqq 0$ a.e. See (2.6). By continuity $J_f \geqq 0$ everywhere except possibly in the singular set S. Thus $K \leqq 0$.

The second half of the theorem is harder: something more than the quasiregularity of f is needed. Fortunately, the result follows immediately from [A–L, Theorem 3]. □

The fact that K is non-positive and may have only isolated zeros means that the p-harmonic surface cannot lie on one side of any of its tangent planes.

The total curvature for the p-harmonic surface

$$T = \{(x, y, u(x, y)) \mid (x, y) \in G\}$$

is by definition

$$\iint_T K d\sigma = \iint_G K\sqrt{1 + |f|^2}\, dx\, dy \tag{5.4}$$

and negative except for planes. (The fact that the total curvature is negative except for planes is a consequence merely of the quasiregularity of the complex gradient $u_x - iu_y$ and hence the same conclusion is valid for a large class of surfaces.) As an application of the Picard theorem for quasiregular mappings we mention the theorem below.

5.5. THEOREM. *Suppose that u is p-harmonic in the whole plane. Then the corresponding p-harmonic surface has total curvature*

$$\iint K d\sigma \leqq -2\pi$$

except when the surface is a plane.

PROOF. The quasiregular mapping $f = u_x - iu_y$ is by assumption defined in the whole plane. It has the Stoïlow representation $f = h \circ \zeta$, h being holomorphic in $\mathbf{C}$ and $\zeta : \mathbf{C} \to \mathbf{C}$ being quasiconformal. Writing $\zeta = \xi + i\eta$ and $h = h_1 + ih_2$, we calculate

$$-\iint_{\mathbf{C}} K\sqrt{1+|f|^2}\,dx\,dy = \iint_{\mathbf{C}} \frac{J_f dx\,dy}{(1+|f|^2)^{3/2}} = \iint_{\mathbf{C}} \frac{|h'(\zeta)|^2 J_\zeta dx\,dy}{(1+|h(\zeta)|^2)^{3/2}}$$

$$= \iint_{\mathbf{C}} \frac{|h'(\zeta)|^2 d\xi\,d\eta}{(1+|h(\zeta)|^2)^{3/2}} \geqq \iint_{\mathbf{C}} \frac{dh_1 dh_2}{(1+|h|^2)^{3/2}} = 2\pi.$$

We used the Picard theorem in evaluating the last integral over the whole plane: h takes all complex values, except possibly one, at least once. (Here one could refine the analysis by taking the multiplicity of h into account.) □

This is again a property close to a related result for minimal surfaces [O, Theorem 9.3, p. 85]. For harmonic surfaces ($p = 2$) a variant of the above result is mentioned in [T 1]. Equality holds for the hyperbolic paraboloid

$$u = \tfrac{1}{2}C(x^2 - y^2),$$

$C \neq 0$ denoting a constant.

Theorem 5.5 holds for any surface defined by a function having quasiregular complex gradient. Theorem 5.3 relies upon a property of the Jacobian which does not hold in this strong version for Jacobians of general quasiregular mappings.

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